Chaotic behavior of disordered nonlinear lattices

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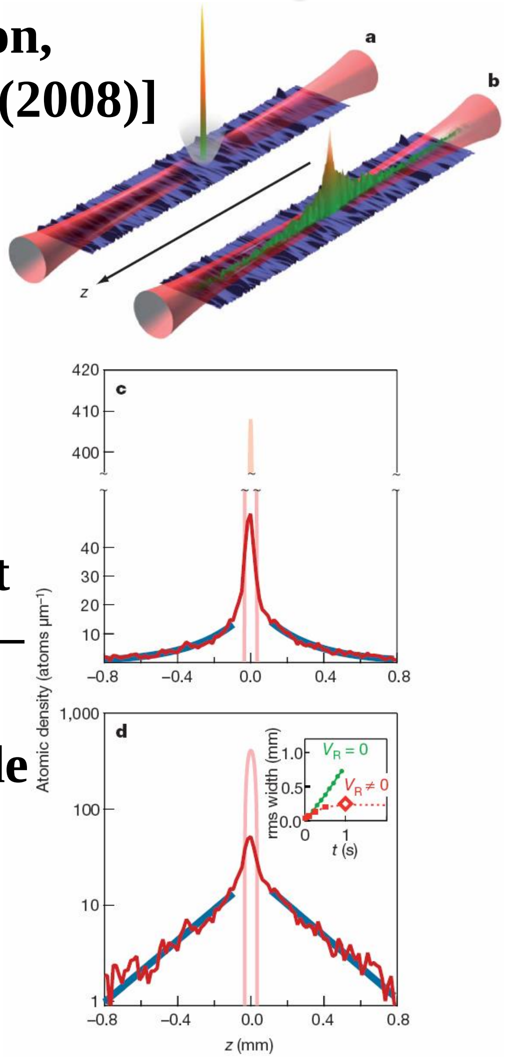
Outline

- **Disordered lattices:**
  - ✓ **The quartic Klein-Gordon (KG) model**
  - ✓ **The disordered nonlinear Schrödinger equation (DNLS)**
  - ✓ **Different dynamical behaviors**
- **Chaotic behavior of the KG model**
  - ✓ **Lyapunov exponents**
  - ✓ **Deviation Vector Distributions**
- **Numerical methods**
  - ✓ **Symplecic Integrators**
  - ✓ **Tangent Map method**
- **Summary**

Interplay of disorder and nonlinearity

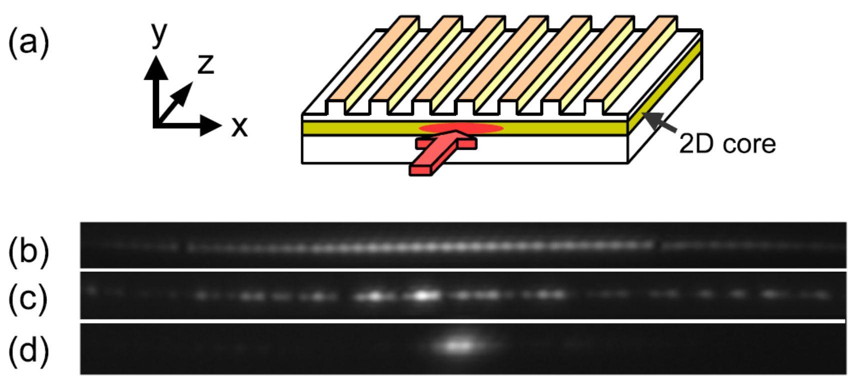
Waves in disordered media – Anderson localization [Anderson, Phys. Rev. (1958)]. Experiments on BEC [Billy et al., Nature (2008)]



Waves in nonlinear disordered media – localization or delocalization?

Theoretical and/or numerical studies [Shepelyansky, PRL (1993) – Molina, Phys. Rev. B (1998) – Pikovsky & Shepelyansky, PRL (2008) – Kopidakis et al., PRL (2008) – Flach et al., PRL (2009) – S. et al., PRE (2009) – Mulansky & Pikovsky, EPL (2010) – S. & Flach, PRE (2010) – Laptyeva et al., EPL (2010) – Mulansky et al., PRE & J.Stat.Phys. (2011) – Bodyfelt et al., PRE (2011) – Bodyfelt et al., IJBC (2011)]

Experiments: propagation of light in disordered 1d waveguide lattices [Lahini et al., PRL (2008)]



The Klein – Gordon (KG) model

$$H_K = \sum_{l=1}^{N} \frac{p_l^2}{2} + \frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W}\left(u_{l+1} - u_l\right)^2$$

with fixed boundary conditions $u_0=p_0=u_{N+1}=p_{N+1}=0$. Typically N=1000.

Parameters: W and the total energy E. $\tilde{\varepsilon}_l$ chosen uniformly from $\left[\frac{1}{2},\frac{3}{2}\right]$.

Linear case (neglecting the term $u_l^4/4$)

Ansatz: $u_l=A_l \exp(i\omega t)$. Normal modes (NMs) $A_{\nu,l}$ - Eigenvalue problem:

$$\lambda A_l = \varepsilon_l A_l - (A_{l+1} + A_{l-1}) \text{ with } \lambda = W\omega^2 - W - 2, \quad \varepsilon_l = W(\tilde{\varepsilon}_l - 1)$$

The discrete nonlinear Schrödinger (DNLS) equation

We also consider the system:

$$H_D = \sum_{l=1}^{N} \varepsilon_l \left|\psi_l\right|^2 + \frac{\beta}{2}\left|\psi_l\right|^4 - \left(\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l\right)$$

where ε_l chosen uniformly from $\left[-\frac{W}{2},\frac{W}{2}\right]$ and β is the nonlinear parameter.

Conserved quantities: The energy and the norm $S = \sum_l \left|\psi_l\right|^2$ of the wave packet.

Distribution characterization

We consider normalized energy distributions in normal mode (NM) space

$$z_\nu \equiv \frac{E_\nu}{\sum_m E_m} \quad \text{with} \quad E_\nu = \frac{1}{2}\left(\dot{A}_\nu^2 + \omega_\nu^2 A_\nu^2\right)$$ **, where A_ν is the amplitude**

of the νth NM (KG) or norm distributions (DNLS).

Second moment: $$m_2 = \sum_{\nu=1}^{N}(\nu - \bar{\nu})^2 z_\nu \quad \text{with} \quad \bar{\nu} = \sum_{\nu=1}^{N} \nu z_\nu$$

Participation number: $$P = \frac{1}{\sum_{\nu=1}^{N} z_\nu^2}$$

measures the number of stronger excited modes in z_ν.

Single mode $P=1$. Equipartition of energy $P=N$.

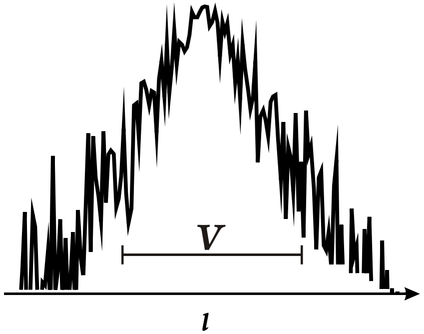
Scales

Linear case: $\omega_\nu^2 \in \left[\frac{1}{2}, \frac{3}{2} + \frac{4}{W}\right]$, **width of the squared frequency spectrum:**

$$\Delta_K = 1 + \frac{4}{W}$$

$$(\Delta_D = W + 4)$$

Localization volume of an eigenstate: $V \sim \frac{1}{\sum_{l=1}^{N} A_{\nu,l}^4}$



Average spacing of squared eigenfrequencies of NMs within the range of a localization volume: $d_K \approx \frac{\Delta_K}{V}$

Nonlinearity induced squared frequency shift of a single site oscillator

$$\delta_l = \frac{3E_l}{2\tilde{\varepsilon}_l} \propto E \qquad (\delta_l = \beta |\psi_l|^2)$$

The relation of the two scales $d_K \leq \Delta_K$ with the nonlinear frequency shift δ_l determines the packet evolution.

Different Dynamical Regimes

Three expected evolution regimes [Flach, Chem. Phys (2010) - S. & Flach, PRE (2010) - Laptyeva et al., EPL (2010) - Bodyfelt et al., PRE (2011)]

Δ: width of the frequency spectrum, d: average spacing of interacting modes,

δ: nonlinear frequency shift.

Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$

Frequency shift is less than the average spacing of interacting modes. NMs are weakly interacting with each other. [Molina, PRB (1998) – Pikovsky, & Shepelyansky, PRL (2008)].

Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2} \longrightarrow m_2 \sim t^{1/3}$

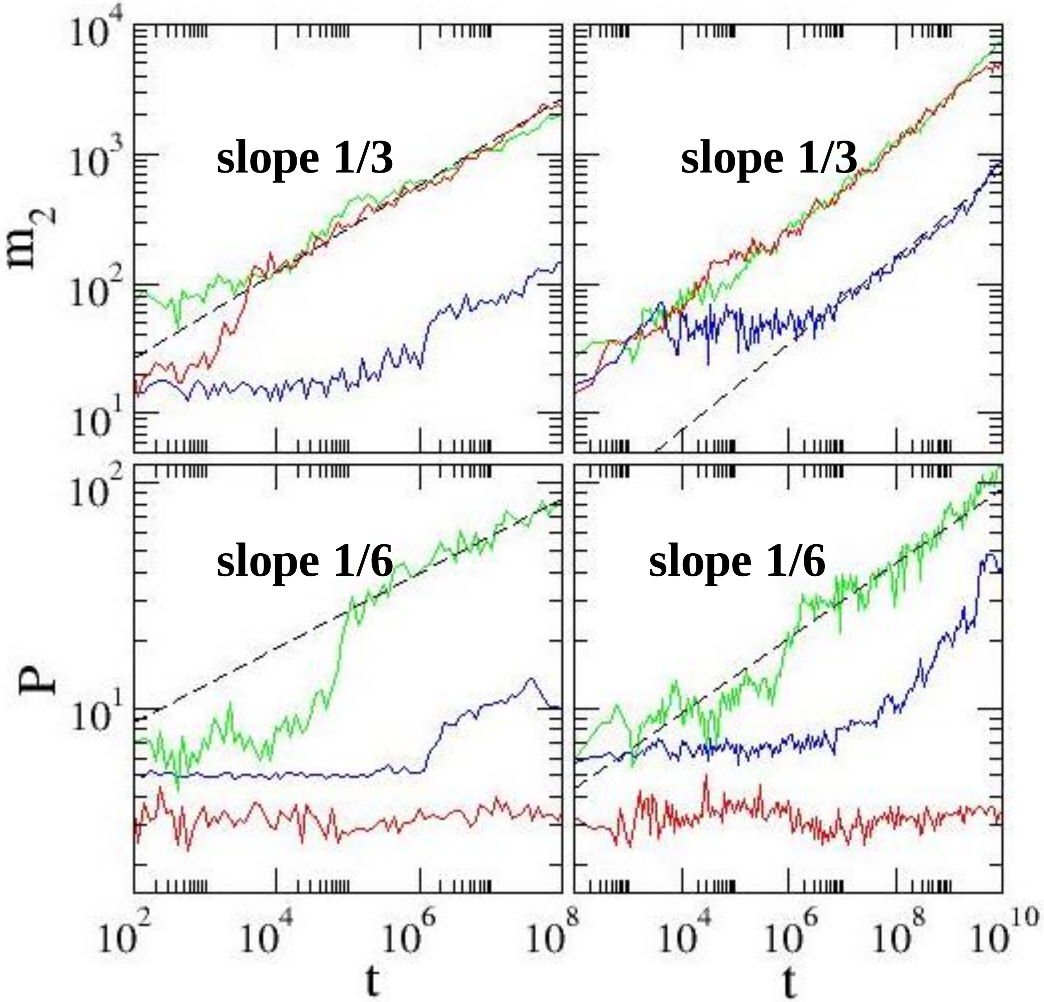
Almost all NMs in the packet are resonantly interacting. Wave packets initially spread faster and eventually enter the weak chaos regime.

Selftrapping Regime: $\delta>\Delta$

Frequency shift exceeds the spectrum width. Frequencies of excited NMs are tuned out of resonances with the nonexcited ones, leading to selftrapping, while a small part of the wave packet subdiffuses [Kopidakis et al., PRL (2008)].

Single site excitations

DNLS **W=4, β= 0.1, 1, 4.5** **KG** **W = 4, E = 0.05, 0.4, 1.5**



No strong chaos regime

In weak chaos regime we averaged the measured exponent α ($m_2 \sim t^{\alpha}$) over 20 realizations:

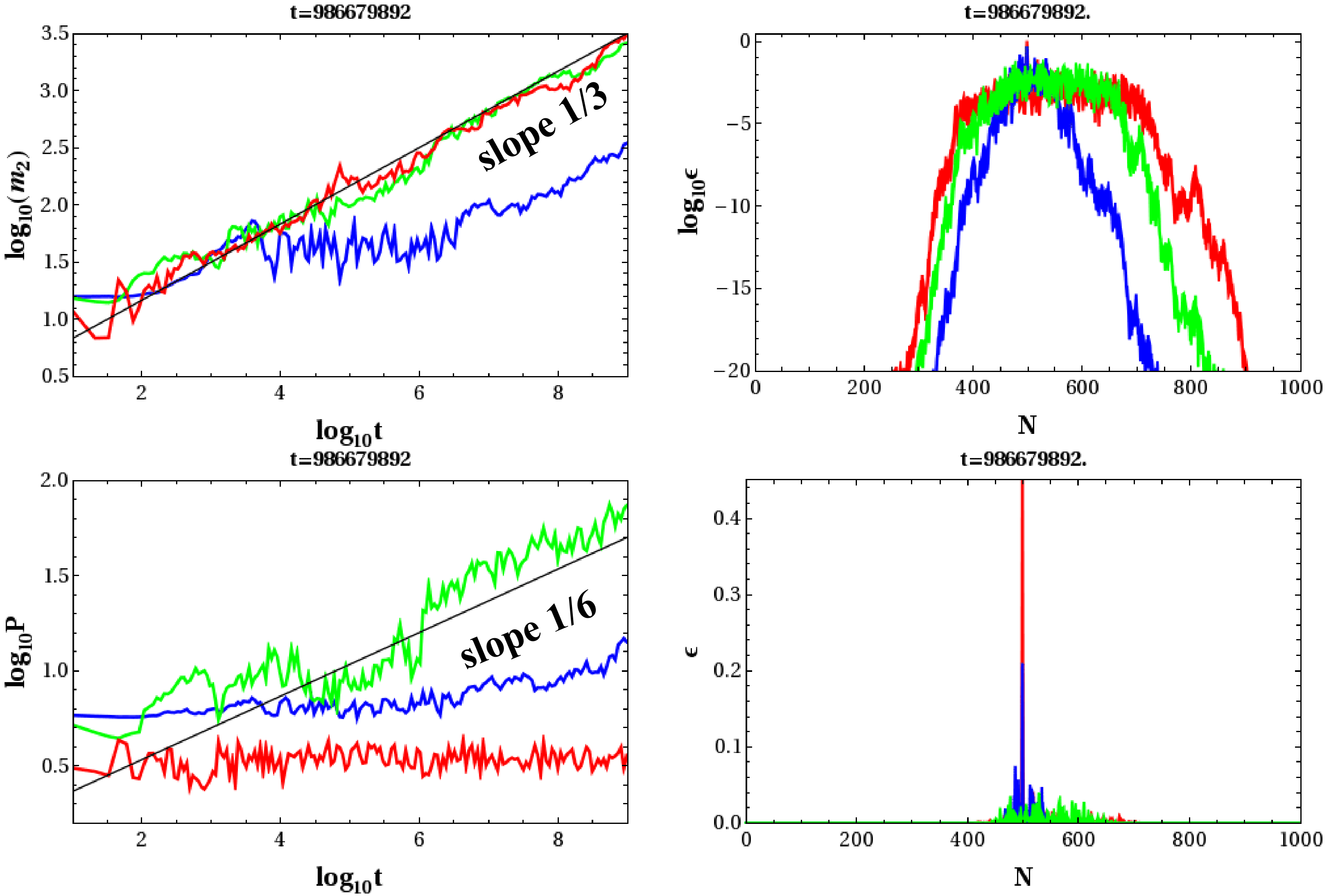
α=0.33±0.05 (KG)

α=0.33±0.02 (DLNS)

Flach et al., PRL (2009)

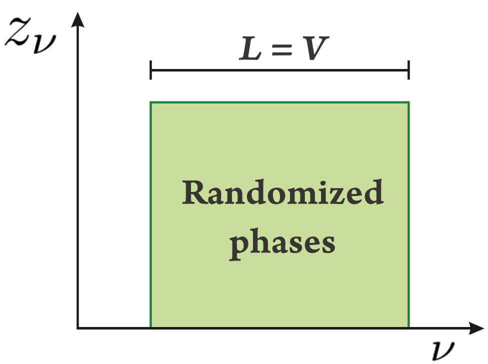
S. et al., PRE (2009)

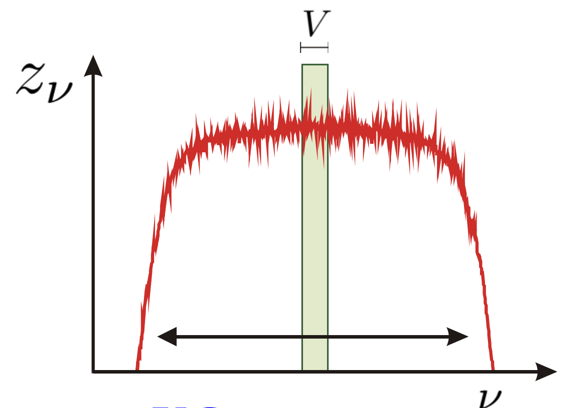
KG: Different spreading regimes



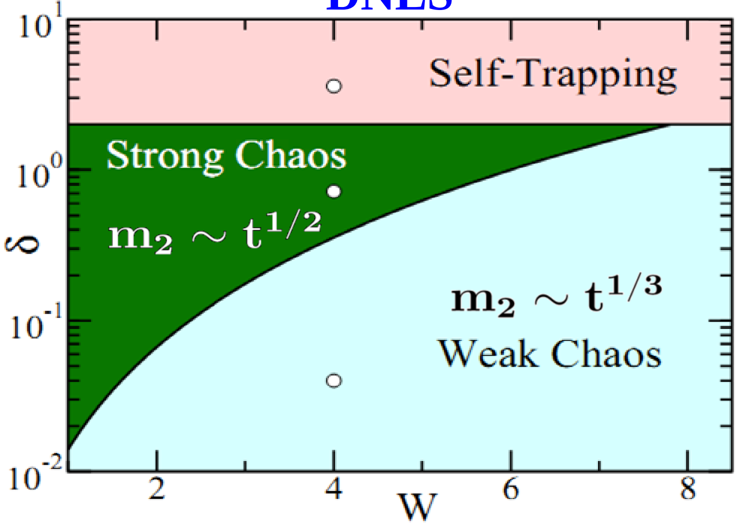
Crossover from strong to weak chaos

We consider compact initial wave packets of width *L=V* [Laptyeva et al., EPL (2010) - Bodyfelt et al., PRE (2011)].

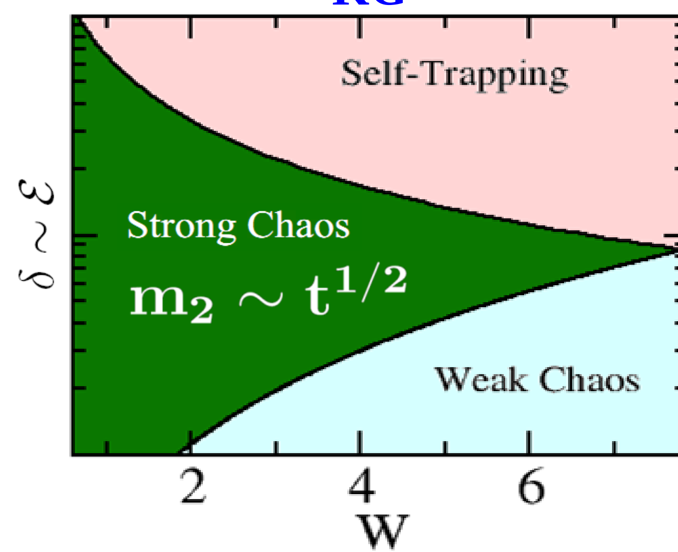




DNLS



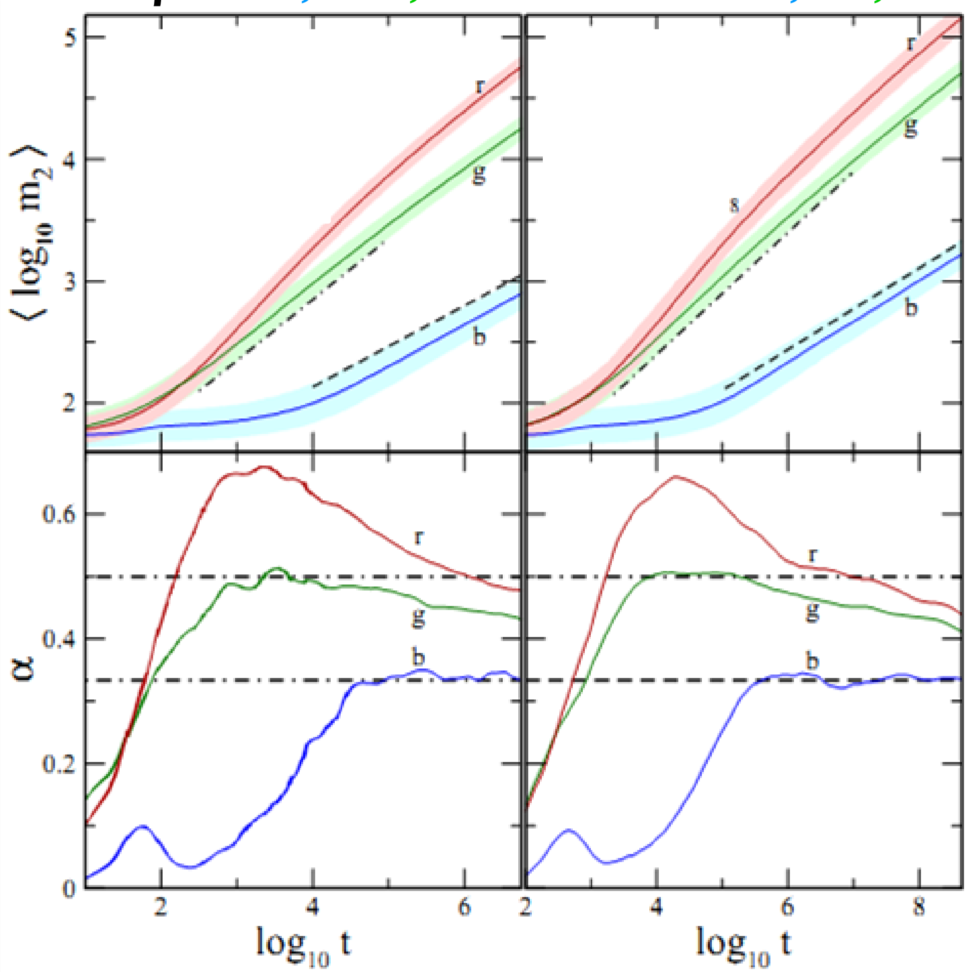
KG



Crossover from strong to weak chaos (block excitations)

DNLS β= 0.04, 0.72, 3.6 **KG E= 0.01, 0.2, 0.75**

W=4



α=1/2

α=1/3

Average over 1000 realizations!

$$\alpha(\log t) = \frac{d\langle \log m_2 \rangle}{d \log t}$$

Laptyeva et al., EPL (2010)

Bodyfelt et al., PRE (2011)

Lyapunov Exponents (LEs)

Roughly speaking, the Lyapunov exponents of a given orbit characterize the mean exponential rate of divergence of trajectories surrounding it.

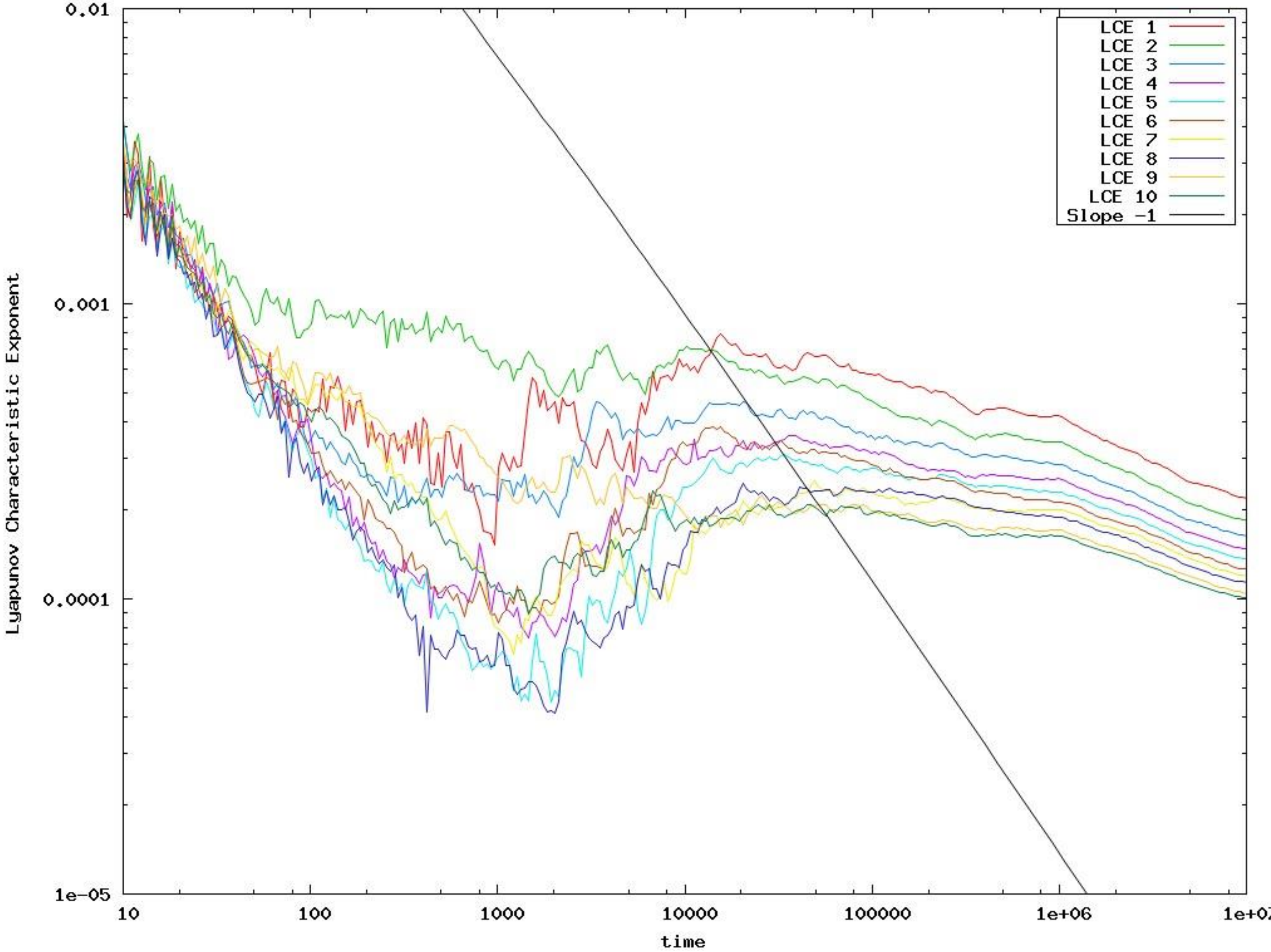
Consider an orbit in the 2N-dimensional phase space with initial condition x(0) and an initial deviation vector from it v(0). Then the mean exponential rate of divergence is:

$$\text{mLCE} = \lambda_1 = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|\vec{v}(t)\|}{\|\vec{v}(0)\|}$$

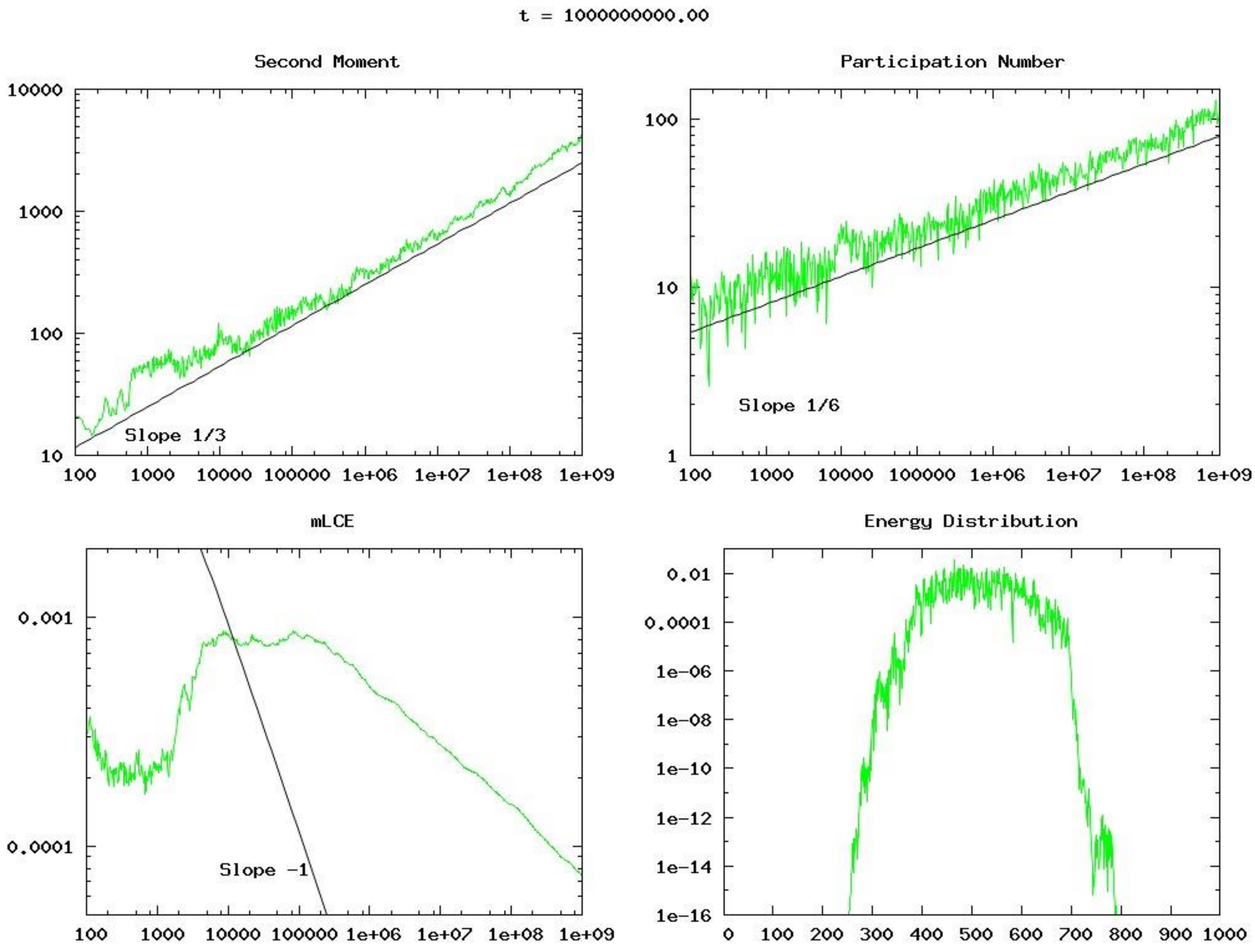
$\lambda_1 = 0 \rightarrow$ **Regular motion** $\propto (t^{-1})$

$\lambda_1 \neq 0 \rightarrow$ **Chaotic motion**

KG: LEs for single site excitations (E=0.4)



KG: Weak Chaos (E=0.4)

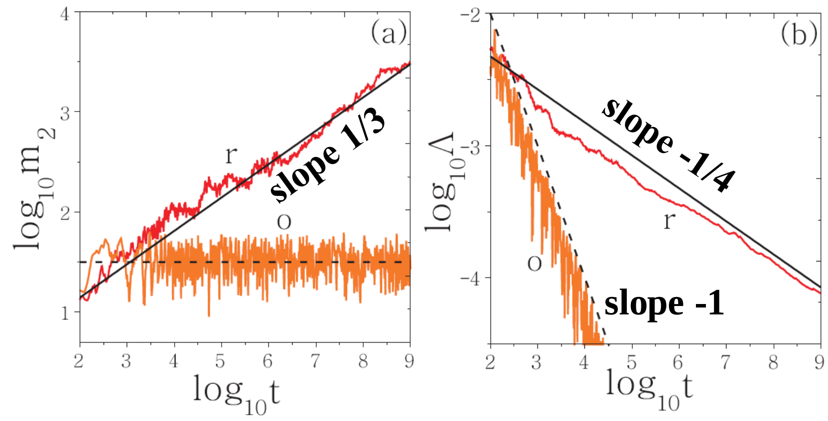


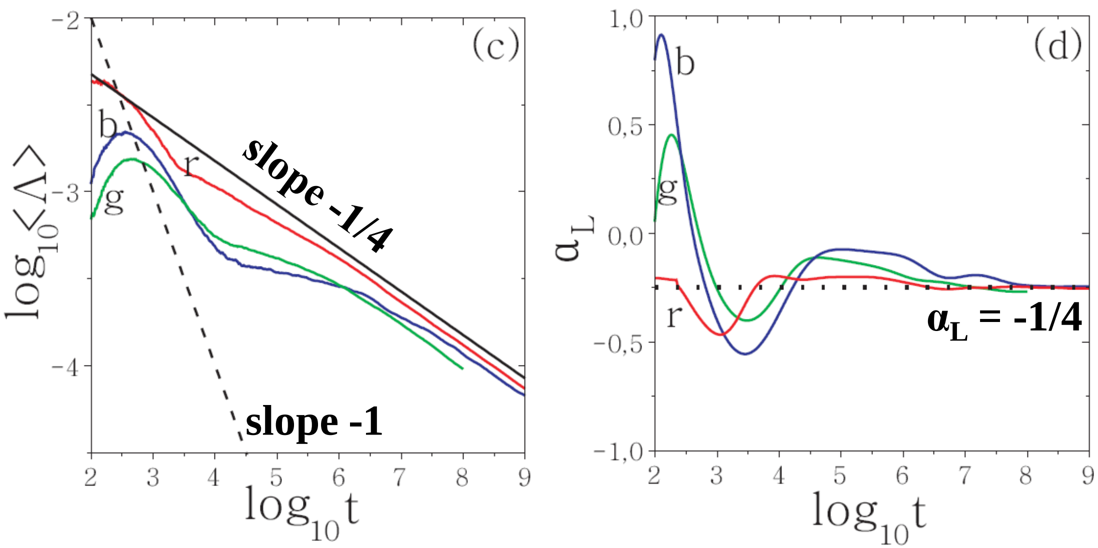
KG: Weak Chaos

Individual runs

Linear case

E=0.4, W=4





$$\alpha_L = \frac{d\left(\log\langle\Lambda\rangle\right)}{d\log t}$$

Average over 50 realizations

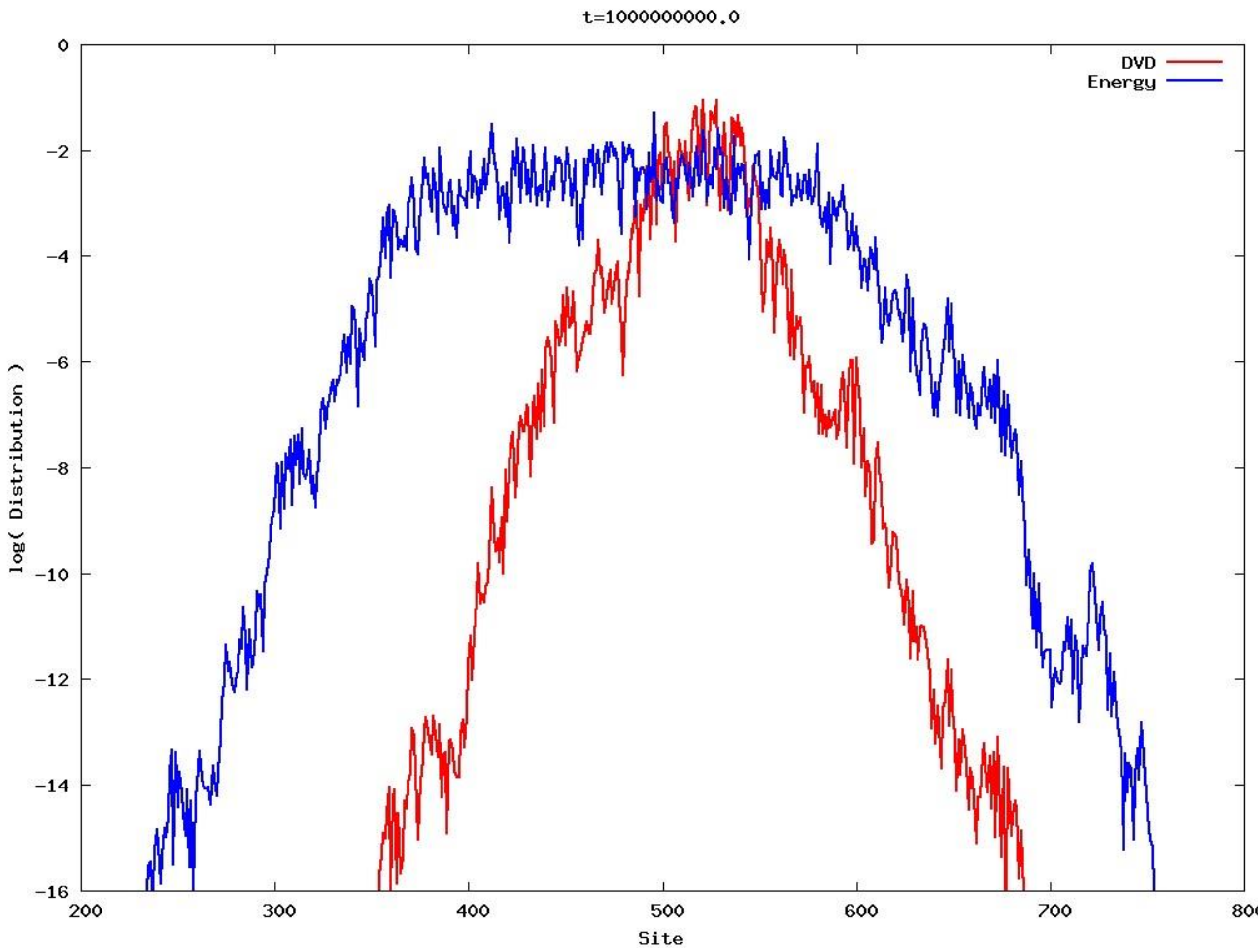
Single site excitation E=0.4, W=4

Block excitation (21 sites) E=0.21, W=4

Block excitation (37 sites) E=0.37, W=3

S. et al. PRL (2013)

Deviation Vector Distributions (DVDs)

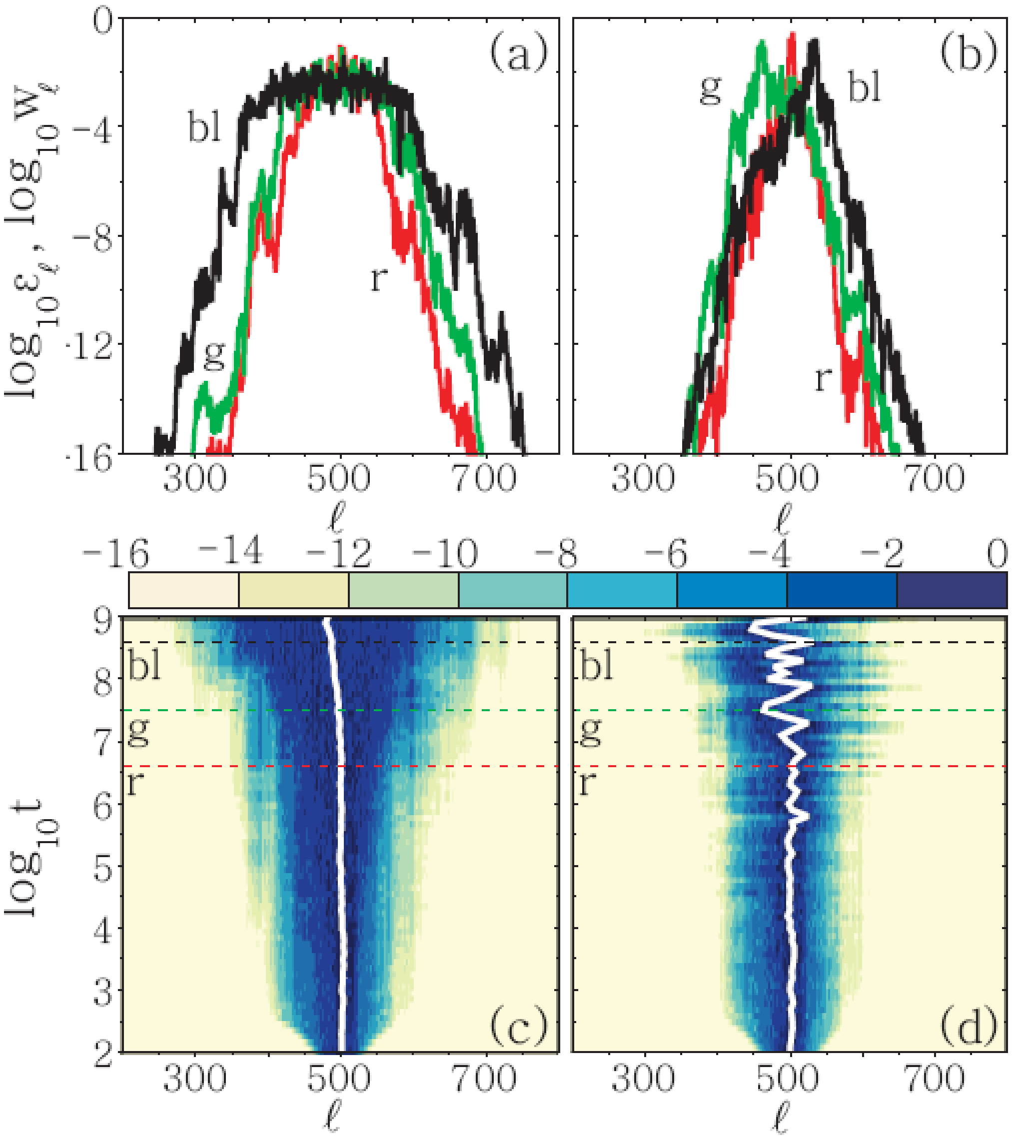


Deviation vector:

v(t)=(δu$_1$(t), δu$_2$(t),…, δu$_N$(t), δp$_1$(t), δp$_2$(t),…, δp$_N$(t))

DVD: $w_l = \dfrac{\delta u_l^2 + \delta p_l^2}{\sum_l \left(\delta u_l^2 + \delta p_l^2\right)}$

Deviation Vector Distributions (DVDs)



Individual run
E=0.4, W=4

Chaotic hot spots meander through the system, supporting a homogeneity of chaos inside the wave packet.

Autonomous Hamiltonian systems

Let us consider an N degree of freedom autonomous Hamiltonian systems of the form:

$$H(\vec{q},\vec{p}) = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + V(\vec{q})$$

As an example, we consider the Hénon-Heiles system:

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

Hamilton equations of motion:

$$\begin{cases} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= y^2 - x^2 - y \end{cases}$$

Variational equations:

$$\begin{cases} \dot{\delta x} &= \delta p_x \\ \dot{\delta y} &= \delta p_y \\ \dot{\delta p_x} &= -(1+2y)\delta x - 2x\delta y \\ \dot{\delta p_y} &= -2x\delta x + (-1+2y)\delta y \end{cases}$$

Symplectic Integrators (SIs)

Formally the solution of the Hamilton equations of motion can be written as:

$$\frac{d\vec{X}}{dt} = \left\{H, \vec{X}\right\} = L_H \vec{X} \Rightarrow \vec{X}(t) = \sum_{n\geq 0} \frac{t^n}{n!} L_H^n \vec{X} = e^{tL_H} \vec{X}$$

where $\vec{X}$ is the full coordinate vector and L_H the Poisson operator:

$$L_H f = \sum_{j=1}^{N} \left\{ \frac{\partial H}{\partial p_j} \frac{\partial f}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial f}{\partial p_j} \right\}$$

If the Hamiltonian H can be split into two integrable parts as H=A+B, a symplectic scheme for integrating the equations of motion from time t to time t+τ consists of approximating the operator $e^{\tau L_H}$ by

$$e^{\tau L_H} = e^{\tau(L_A + L_B)} = \prod_{i=1}^{j} e^{c_i \tau L_A} e^{d_i \tau L_B} + O(\tau^{n+1})$$

for appropriate values of constants c_i, d_i. This is an integrator of order n.

So the dynamics over an integration time step τ is described by a series of successive acts of Hamiltonians A and B.

Symplectic Integrator $SABA_2C$

The operator $e^{\tau L_H}$ can be approximated by the symplectic integrator [Laskar & Robutel, Cel. Mech. Dyn. Astr. (2001)]:

$$SABA_2 = e^{c_1 \tau L_A}\; e^{d_1 \tau L_B}\; e^{c_2 \tau L_A}\; e^{d_1 \tau L_B}\; e^{c_1 \tau L_A}$$

with $c_1 = \frac{1}{2} - \frac{\sqrt{3}}{6},\; c_2 = \frac{\sqrt{3}}{3},\; d_1 = \frac{1}{2}.$

The integrator has only small positive steps and its error is of order 2.

In the case where *A* is quadratic in the momenta and *B* depends only on the positions the method can be improved by introducing a corrector *C*, having a small negative step:

$$C = e^{-\tau^3 \frac{c}{2} L_{\{\{A,B\},B\}}}$$

with $c = \frac{2 - \sqrt{3}}{24}.$

Thus the full integrator scheme becomes: $SABAC_2 = C\,(SABA_2)\,C$ and its error is of order 4.

Tangent Map (TM) Method

Any symplectic integration scheme used for solving the Hamilton equations of motion, which involves the act of Hamiltonians A and B, can be extended in order to integrate simultaneously the variational equations **[S. & Gerlach, PRE (2010) – Gerlach & S., Discr. Cont. Dyn. Sys. (2011) – Gerlach et al., IJBC (2012)].**

The Hénon-Heiles system can be split as: $A = \frac{1}{2}(p_x^2 + p_y^2) \quad B = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$

$$\begin{aligned}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y \\
\dot{\delta x} &= \delta p_x \\
\dot{\delta y} &= \delta p_y \\
\dot{\delta p}_x &= -(1+2y)\delta x - 2x\delta y \\
\dot{\delta p}_y &= -2x\delta x + (-1+2y)\delta y
\end{aligned}$$

$A(\vec{p})$ →

$$\left.\begin{aligned}
\dot{x} &= p_x \\
\dot{y} &= p_y \\
\dot{p}_x &= 0 \\
\dot{p}_y &= 0 \\
\dot{\delta x} &= \delta p_x \\
\dot{\delta y} &= \delta p_y \\
\dot{\delta p}_x &= 0 \\
\dot{\delta p}_y &= 0
\end{aligned}\right\} \Rightarrow \frac{d\vec{u}}{dt} = L_{AV}\vec{u} \quad \Rightarrow \quad e^{\tau L_{AV}} : \left\{\begin{aligned}
x' &= x + p_x \tau \\
y' &= y + p_y \tau \\
px' &= p_x \\
py' &= p_y \\
\delta x' &= \delta x + \delta p_x \tau \\
\delta y' &= \delta y + \delta p_y \tau \\
\delta p_x' &= \delta p_x \\
\delta p_y' &= \delta p_y
\end{aligned}\right.$$

$B(\vec{q})$ →

$$\left.\begin{aligned}
\dot{x} &= 0 \\
\dot{y} &= 0 \\
\dot{p}_x &= -x - 2xy \\
\dot{p}_y &= y^2 - x^2 - y \\
\dot{\delta x} &= 0 \\
\dot{\delta y} &= 0 \\
\dot{\delta p}_x &= -(1+2y)\delta x - 2x\delta y \\
\dot{\delta p}_y &= -2x\delta x + (-1+2y)\delta y
\end{aligned}\right\} \Rightarrow \frac{d\vec{u}}{dt} = L_{BV}\vec{u} \quad \Rightarrow \quad e^{\tau L_{BV}} : \left\{\begin{aligned}
x' &= x \\
y' &= y \\
p_x' &= p_x - x(1+2y)\tau \\
p_y' &= p_y + (y^2 - x^2 - y)\tau \\
\delta x' &= \delta x \\
\delta y' &= \delta y \\
\delta p_x' &= \delta p_x - [(1+2y)\delta x + 2x\delta y]\tau \\
\delta p_y' &= \delta p_y + [-2x\delta x + (-1+2y)\delta y]\tau
\end{aligned}\right.$$

The KG model

We apply the SABAC$_2$ integrator scheme to the KG Hamiltonian by using the splitting:

$$H_K = \sum_{l=1}^{N} \left(\underbrace{\frac{p_l^2}{2}}_{A} + \underbrace{\frac{\tilde{\varepsilon}_l}{2} u_l^2 + \frac{1}{4} u_l^4 + \frac{1}{2W} \left(u_{l+1} - u_l \right)^2}_{B} \right)$$

$$e^{\tau L_A}: \begin{cases} u_l' = p_l \tau + u_l \\ p_l' = p_l, \end{cases}$$

$$e^{\tau L_B}: \begin{cases} u_l' = u_l \\ p_l' = \left[-u_l(\tilde{\epsilon}_l + u_l^2) + \frac{1}{W}(u_{l-1} + u_{l+1} - 2u_l) \right] \tau + p_l, \end{cases}$$

with a corrector term which corresponds to the Hamiltonian function:

$$C = \left\{ \left\{ A, B \right\}, B \right\} = \sum_{l=1}^{N} \left[u_l (\tilde{\varepsilon}_l + u_l^2) - \frac{1}{W} (u_{l-1} + u_{l+1} - 2u_l) \right]^2 .$$

The DNLS model

A 2nd order SABA Symplectic Integrator with 5 steps, combined with approximate solution for the *B* part (Fourier Transform): SIFT²

$$H_D = \sum_l \varepsilon_l |\psi_l|^2 + \frac{\beta}{2}|\psi_l|^4 - \left(\psi_{l+1}\psi_l^* + \psi_{l+1}^*\psi_l\right), \qquad \psi_l = \frac{1}{\sqrt{2}}\left(q_l + ip_l\right)$$

$$H_D = \sum_l \left(\underbrace{\frac{\varepsilon_l}{2}\left(q_l^2 + p_l^2\right) + \frac{\beta}{8}\left(q_l^2 + p_l^2\right)^2}_{A} \underbrace{- q_n q_{n+1} - p_n p_{n+1}}_{B} \right)$$

$$e^{\tau L_A}: \begin{cases} q_l' = q_l \cos(\alpha_l \tau) + p_l \sin(\alpha_l \tau), \\ p_l' = p_l \cos(\alpha_l \tau) - q_l \sin(\alpha_l \tau), \end{cases}$$

$$\alpha_l = \epsilon_l + \beta(q_l^2 + p_l^2)/2$$

$$e^{\tau L_B}: \begin{cases} \varphi_q = \sum_{m=1}^{N} \psi_m e^{2\pi i q(m-1)/N} \\ \varphi_q' = \varphi_q e^{2i\cos(2\pi(q-1)/N)\tau} \\ \psi_l' = \frac{1}{N}\sum_{q=1}^{N} \varphi_q' e^{-2\pi i l(q-1)/N} \end{cases}$$

The DNLS model

Symplectic Integrators produced by Successive Splits (SS)

$$H_D = \sum_l \left(\underbrace{\frac{\varepsilon_l}{2}\left(q_l^2 + p_l^2\right) + \frac{\beta}{8}\left(q_l^2 + p_l^2\right)^2}_{A} \underbrace{- q_n q_{n+1} - p_n p_{n+1}}_{B} \right)$$

A →
$$\begin{cases} q_l' = q_l \cos(\alpha_l \tau) + p_l \sin(\alpha_l \tau), \\ p_l' = p_l \cos(\alpha_l \tau) - q_l \sin(\alpha_l \tau), \end{cases}$$

B_1 →
$$\begin{cases} q_l' = q_l, \\ p_l' = p_l + (q_{l-1} + q_{l+1})\tau \end{cases}$$

B_2 →
$$\begin{cases} p_l' = p_l, \\ q_l' = q_l - (p_{l-1} + p_{l+1})\tau \end{cases}$$

Using the SABA$_2$ integrator we get a 2nd order integrator with 13 steps, SS^2:

$$SS^2 = e^{\left[\frac{(3-\sqrt{3})}{6}\tau\right]L_A} e^{\frac{\tau}{2}L_B} e^{\frac{\sqrt{3}\tau}{3}L_A} e^{\frac{\tau}{2}L_B} e^{\left[\frac{(3-\sqrt{3})}{6}\tau\right]L_A}$$

$$\tau' = \tau / 2$$

$$e^{\frac{\tau}{2}L_B} \rightarrow e^{\left[\frac{(3-\sqrt{3})}{6}\tau'\right]L_{B_1}} e^{\frac{\tau'}{2}L_{B_2}} e^{\frac{\sqrt{3}\tau'}{3}L_{B_1}} e^{\frac{\tau'}{2}L_{B_2}} e^{\left[\frac{(3-\sqrt{3})}{6}\tau'\right]L_{B_1}}$$

$$e^{\frac{\tau}{2}L_B} \rightarrow e^{\left[\frac{(3-\sqrt{3})}{6}\tau'\right]L_{B_1}} e^{\frac{\tau'}{2}L_{B_2}} e^{\frac{\sqrt{3}\tau'}{3}L_{B_1}} e^{\frac{\tau'}{2}L_{B_2}} e^{\left[\frac{(3-\sqrt{3})}{6}\tau'\right]L_{B_1}}$$

Three part split symplectic integrators for the DNLS model

Three part split symplectic integrator of order 2, with 5 steps: ABC^2

$$H_D = \sum_l \underbrace{\left(\frac{\varepsilon_l}{2}\left(q_l^2 + p_l^2\right) + \frac{\beta}{8}\left(q_l^2 + p_l^2\right)^2 \right.}_{A} \underbrace{- q_n q_{n+1}}_{B} \underbrace{\left. - p_n p_{n+1} \right)}_{C}$$

$$ABC^2 = e^{\frac{\tau}{2}L_A} e^{\frac{\tau}{2}L_B} e^{\tau L_C} e^{\frac{\tau}{2}L_B} e^{\frac{\tau}{2}L_A}$$

This low order integrator has already been used by e.g. Chambers, MNRAS (1999) – Goździewski et al., MNRAS (2008).

Composition Methods: 4^{th} order SIs

Starting from any 2^{nd} order symplectic integrator S^{2nd}, we can construct a 4^{th} order integrator S^{4th} using the composition method proposed by Yoshida [Phys. Lett. A (1990)]:

$$S^{4th}(\tau) = S^{2nd}(x_1\tau) \times S^{2nd}(x_0\tau) \times S^{2nd}(x_1\tau), \qquad x_0 = -\frac{2^{1/3}}{2-2^{1/3}}, \qquad x_1 = \frac{1}{2-2^{1/3}}$$

In this way, starting with the 2^{nd} order integrators SS^2, $SIFT^2$ and ABC^2 we construct the 4^{th} order integrators:

SS^4 with 37 steps **$SIFT^4$ with 13 steps** **$ABC^4_{[Y]}$ with 13 steps**

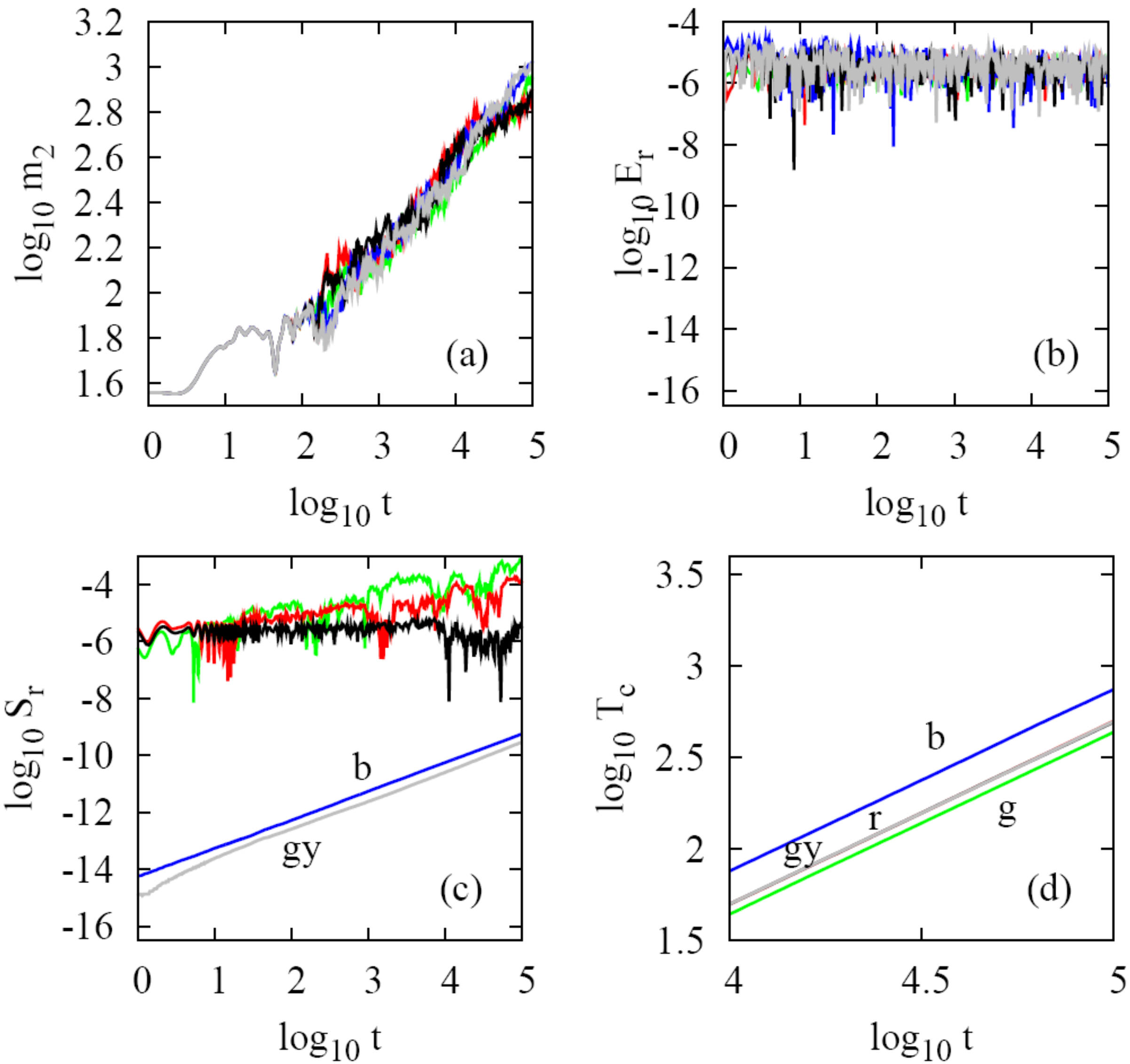
Composition method proposed by Suzuki [Phys. Lett. A (1990)]:

$$S^{4th}(\tau) = S^{2nd}(p_2\tau) \times S^{2nd}(p_2\tau) \times S^{2nd}((1-4p_2)\tau) \times S^{2nd}(p_2\tau) \times S^{2nd}(p_2\tau)$$

$$p_2 = \frac{1}{4-4^{1/3}}, \qquad 1-4p_2 = -\frac{4^{1/3}}{4-4^{1/3}}$$

Starting with the 2^{nd} order integrators ABC^2 we construct the 4^{th} order integrator: $ABC^4_{[S]}$ with 21 steps.

4^{th} order integrators: Numerical results (I)



$SIFT^4$ $\tau=0.125$

$SIFT^2$ $\tau=0.05$

$ABC^4_{[S]}$ $\tau=0.1$

SS^4 $\tau=0.1$

$ABC^4_{[Y]}$ $\tau=0.05$

E_r: relative energy error

S_r: relative norm error

T_c: CPU time (sec)

S. et al., Phys. Lett. A (2014)

Summary

- **We presented three different dynamical behaviors for wave packet spreading in 1d nonlinear disordered lattices:**
  - ✓ **Weak Chaos Regime: $\delta<d$, $m_2 \sim t^{1/3}$**
  - ✓ **Intermediate Strong Chaos Regime: $d<\delta<\Delta$, $m_2 \sim t^{1/2} \longrightarrow m_2 \sim t^{1/3}$**
  - ✓ **Selftrapping Regime: $\delta>\Delta$**
- **Lyapunov exponent computations show that:**
  - ✓ **Chaos not only exists, but also persists.**
  - ✓ **Slowing down of chaos does not cross over to regular dynamics.**
  - ✓ **Chaotic hot spots meander through the system, supporting a homogeneity of chaos inside the wave packet.**
- **Our results suggest that Anderson localization is eventually destroyed by nonlinearity, since spreading does not show any sign of slowing down.**
- **We emphasized the use of symplectic schemes based on 3 part split of the Hamiltonian (ABC methods) for the integration of the DNLS model.**

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Thank you for your attention